

On Dirac Zero Modes in Hyperdiamond Model

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January 14, 2013

Abstract

Using the $SU(5)$ symmetry of the $4D$ hyperdiamond and results on the study of $4D$ graphene given in [10], we engineer a class of $4D$ lattice QCD fermions whose Dirac operators have two zero modes. We show that generally the zero modes of the Dirac operator in hyperdiamond fermions are captured by a tensor Ω_μ^l with 4×5 complex components linking the Euclidean $SO(4)$ vector μ ; and the 5-dimensional representation of $SU(5)$. The *Boriçi-Creutz* (BC) and the *Karsten-Wilzeck* (KW) models as well as their Dirac zero modes are rederived as particular realizations of Ω_μ^l . Other features are also given.

Keywords: Lattice QCD, *Boriçi-Creutz* and *Karsten-Wilzeck* models, $4D$ hyperdiamond, $4D$ graphene, $SU(5)$ Symmetry.

1 Introduction

One of the highly important things in lattice QCD is the need to have a fermion action with a Dirac operator \mathcal{D} having two zero modes at the points K and K' of the reciprocal space; so that they could be interpreted as the two light quarks, up and down. In this regards, *Michael Creutz* proposed few years ago an interesting $4D$ lattice fermion action [1] by extending the dispersion energy relation of $2D$ graphene [2, 3] to $4D$ quaternions. The *Creutz* fermions, which exhibit very useful properties for dealing with lattice QCD , was

further developed by *Artan Boriçi* [4] leading afterwards to the so called *Boriçi-Creutz* (*BC*) fermions [5, 6]. This is a simple $4D$ fermion lattice model using a 4-component Dirac spinor $\Psi_{\mathbf{r}}$ as well as a particular linear combination of the 4×4 matrices γ_{μ} . *BC* lattice model has too particularly chiral symmetry, preventing mass renormalization, and a Dirac operator \mathcal{D}_{BC} with two zero modes K_{BC} and K'_{BC} that describe a degenerate doublet of quarks. The engineering of these two zeros is mainly due to mimicking $2D$ graphene as well as to a nice trick relying on using a "complexification" of the Dirac matrices type $\Upsilon_{\mu} = \gamma_{\mu} + i\gamma'_{\mu}$ and the remarkable linear combination $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. This way of doing let suspect that the *BC* model hides a more fundamental property that controls, amongst others, the engineering of Dirac operator zeros.

In this paper, we contribute to this matter by proposing a fermionic $4D$ lattice *QCD*-like model based on $SU(5)$ symmetry and containing *BC* and *KW* [7, 8] fermions as particular cases. In our proposal the $4D$ lattice, denoted as \mathbb{L}_4 , is given by the hyperdiamond generated by the 4 simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $SU(5)$ [9]. Moreover, each site \mathbf{r}_A of \mathbb{L}_4 has 5 first nearest neighbors $(\mathbf{r}_A + a\boldsymbol{\lambda}_l)$ forming a 5-dimensional representation of the $SU(5)$ symmetry.

The proposed free massless fermion lattice action is given by

$$\mathcal{S} \sim \frac{i}{4a} \sum_{\mathbf{r}} \sum_{\mu=1}^4 \left(\sum_{l=0}^4 \bar{\Psi}_{\mathbf{r}} (\gamma^{\mu} \Omega_{\mu}^l) \Psi_{\mathbf{r}+a\boldsymbol{\lambda}_l} + \sum_{l=0}^4 \bar{\Psi}_{\mathbf{r}+a\boldsymbol{\lambda}_l} (\gamma^{\mu} \bar{\Omega}_{\mu}^l) \Psi_{\mathbf{r}} \right) \quad (1.1)$$

where $\Psi_{\mathbf{r}}$ lives on \mathbb{L}_4 and where the 5 vectors $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ stand for the weight vectors of the 5-dimensional representation of $SU(5)$ satisfying the basic property $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. This identity has a physical interpretation in terms of conservation of total momenta at each site \mathbf{r} ; and has as well a strong link with the linear combination $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 - 2\Gamma = 0$ used in *BC* model. The distance $a \|\boldsymbol{\lambda}_l\| = a \frac{2\sqrt{5}}{5}$ is the parameter d of the lattice.

Notice that our lattice fermion model depends on the complex tensor Ω_{μ}^l whose features will be studied here in details. It is a kind of a complex 4×5 rectangular matrix that plays a quite similar role to the matrix combinations $\gamma_{\mu} + i\gamma'_{\mu}$, $\gamma_{\mu} + i\gamma_4(1 - \delta_{\mu 4})$ and $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ used in the *BC* and *KW* models [5, 6]. In the case of free *BC* fermions, we will show that the corresponding $(\Omega_{\mu}^l)_{BC}$ tensor reads as

$$\begin{pmatrix} 2 & 1+2i & -1 & -1 & -1 \\ 2 & -1 & 1+2i & -1 & -1 \\ 2 & -1 & -1 & 1+2i & -1 \\ 2 & -1 & -1 & -1 & 1+2i \end{pmatrix}, \quad (1.2)$$

and for the KW ones, the associated $(\Omega_\mu^l)_{KW}$ is given by

$$\begin{pmatrix} 0 & i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 2 & 1 & 1 & -2 & i-2 \end{pmatrix}. \quad (1.3)$$

Notice also that, generally speaking, Ω_μ^l transforms in the bi-fundamental of $SO(4) \times SU(5)$; and should not be a tensor product; i.e $\Omega_\mu^l \neq \xi_\mu \times \zeta^l$ in order to recover the continuum limit. This Ω_μ^l links then the $SO(4)$ and $SU(5)$ fundamental representations respectively captured by the indices $\mu = 1, 2, 3, 4$ and $l = 0, 1, 2, 3, 4$. It allows as well to implement quark-quark-gluon interactions by letting it to depend on positions \mathbf{r} .

The tensor Ω_μ^l has another remarkable feature that we are interested in here; it plays a crucial role in the engineering of lattice Dirac operator with a generic number N of zeros; including the interesting case $N = 2$. Splitting Ω_μ^l into real and imaginary parts as $R_\mu^l + iJ_\mu^l$, it is clear that the set of real parameters captured by the rectangular matrix R_μ^l , and similarly for J_μ^l , is given by the real 20-dimensional moduli space

$$\mathfrak{R}_{20} = \frac{SO(4, 5)}{SO(4) \times SO(5)}, \quad (1.4)$$

so that the total moduli space of (1.1) is given by the complexification of \mathfrak{R}_{20} namely $\frac{U(4,5)}{U(4) \times U(5)}$ with dimension $\dim_R U(4, 5) - \dim_R U(4) - \dim_R U(5) = 40$. As such, the zero modes of the lattice "Dirac" operator $\mathcal{D}_{k,\Omega} \sim \gamma^\mu (\Omega_\mu^l e^{iak \cdot \lambda_l} + \bar{\Omega}_\mu^l e^{-iak \cdot \lambda_l})$ of the above lattice action (1.1) correspond to some particular tensors $(\Omega_\mu^l)_{N\text{-zeros}}$; and so would live on subspaces of $\mathfrak{R}_{20} \times \mathfrak{R}_{20}$. From this moduli space view, operators $\mathcal{D}_{k,\Omega}$ that have two zero modes have tensors type $(\Omega_\mu^l)_{2\text{-zeros}}$ which require the fixing $(4+4)$ real moduli; and therefore would live on real 32-dimensional subspaces \mathfrak{M}_{32} of the moduli subspace. Typical examples of \mathfrak{M}_{32} is given by the particular space

$$\frac{U(4, 4)}{U(4) \times U(4)}, \quad (1.5)$$

or more generally by $\mathfrak{R}_{20-n} \times \mathfrak{R}_{20-m}$ with $n+m = 8$. In addition to the previous BC and KW examples with two zeros, one can engineer others by solving appropriate constraint relations or directly by making wise choices based using symmetries. This analysis is developed in sub-section 4.2 from which we learn that the real $(R_\mu^l)_{BC}$ and imaginary $(J_\mu^l)_{BC}$ parts of the BC complex tensor $(\Omega_\mu^l)_{BC}$ should obey the constraint relations

$$\sum_{\nu=1}^4 (R_\mu^\nu)_{BC} = - (R_\mu^0)_{BC} \quad , \quad \sum_{\nu=1}^4 (J_\mu^\nu)_{BC} = (R_\mu^0)_{BC} \quad ,$$

Similar relations can be also written down for the tensors $(R_\mu^l)_{KW}$ and $(J_\mu^l)_{KW}$ in the case of the KW model; see eqs (4.45).

The presentation of this paper is as follows: In section 2, we recall briefly some results on the BC and KW models; in particular the engineering of the two zeros of the corresponding Dirac operator. We also give extensions of these models where the linear combination $\Gamma_5 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is interpreted in terms of a fifth direction; the four others are $\Gamma_\mu = \gamma_\mu + i\gamma'_\mu$. These extensions may also be understood as an indication for the existence of the tensor Ω_μ^l that captures data on the zeros of Dirac operators. In section 3, we develop our proposal; and show, amongst others, that the Dirac operator following from (1.1) can also have two zero modes by making wise choices of Ω_μ^l . In section 4, we re-derive the BC and KW theories, and their extensions using Γ_5 term, from (1.1). In section 5, we give a conclusion and a comment on gauged (1.1).

2 BC/KW fermions and beyond

First we study the two zero modes K_{BC} and K'_{BC} of the Dirac operator \mathcal{D}_{BC} of the BC fermions; then we give those K_{KW} and K'_{KW} of the Dirac operator \mathcal{D}_{KW} of KW model. We also use this section to give an extension of BC and KW models where the term $\bar{\Psi}_\mathbf{r}\Gamma\Psi_\mathbf{r}$ is modified as $\bar{\Psi}_\mathbf{r}\Gamma^5\Psi_{\mathbf{r}+a\mathbf{5}}$.

The analysis given in this section is useful to fix the ideas; but also for later use since BC , KW models and their extensions can be re-derived from the lattice action (1.1) by using wise choices of the tensor Ω_μ^l .

2.1 BC fermions: the free model

Following [5, 6] and using the 4-component Dirac spinors $\Psi_\mathbf{r} = (\phi_\mathbf{r}^a, \bar{\chi}_\mathbf{r}^{\dot{a}})$, the lattice action of *free* BC fermions reads in the position space, by dropping mass term m_0 , as follows:

$$\mathcal{S}_{BC} \sim \frac{1}{2a} \sum_{\mathbf{r}} \left(\sum_{\mu=1}^4 \bar{\Psi}_\mathbf{r} \Upsilon^\mu \Psi_{\mathbf{r}+a\boldsymbol{\mu}} - \sum_{\mu=1}^4 \bar{\Psi}_{\mathbf{r}+a\boldsymbol{\mu}} \bar{\Upsilon}^\mu \Psi_\mathbf{r} \right) - \frac{2i}{a} \sum_{\mathbf{r}} \bar{\Psi}_\mathbf{r} \Gamma \Psi_\mathbf{r} \quad (2.1)$$

where, for simplicity, we have dropped out gauge interactions; and where $\Upsilon^\mu = \gamma^\mu + i\gamma'^\mu$; which is a kind of complexification of the Dirac matrices.

Moreover, the matrix Γ appearing in the last term is a 4×4 matrix linked to γ^μ , γ'^μ as

follows:

$$\begin{aligned}\gamma'^\mu &= \Gamma - \gamma^\mu \quad , \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \\ 2\Gamma &= \sum_{\mu=1}^4 \gamma^\mu \quad , \quad \gamma^\mu + i\gamma'^\mu = \Upsilon^\mu\end{aligned}\tag{2.2}$$

Mapping (2.1) to the reciprocal space, we have

$$\mathcal{S}_{BC} \sim \sum_{\mathbf{k}} \bar{\Psi}_{\mathbf{k}} \mathcal{D}_{BC} \Psi_{\mathbf{k}}\tag{2.3}$$

where the massless Dirac operator \mathcal{D}_{BC} is given by

$$\begin{aligned}\mathcal{D}_{BC} = & +\frac{1}{2a} (\Upsilon_\mu - \bar{\Upsilon}_\mu) \cos(ak_\mu) \\ & +\frac{i}{2a} (\Upsilon_\mu + \bar{\Upsilon}_\mu) \sin(ak_\mu) - \frac{2i}{a} \Gamma \quad .\end{aligned}\tag{2.4}$$

Upon using $\Upsilon_\mu + \bar{\Upsilon}_\mu = 2\gamma_\mu$ and $\Upsilon_\mu - \bar{\Upsilon}_\mu = 2i\gamma'_\mu$, we can be put \mathcal{D}_{BC} in the form

$$\mathcal{D}_{BC} = D_{\mathbf{k}} + \bar{D}_{\mathbf{k}} - \frac{2i}{a} \Gamma\tag{2.5}$$

with

$$D_{\mathbf{k}} = \frac{i}{a} \left(\sum_{\mu=1}^4 \gamma^\mu \sin ak_\mu \right) \quad , \quad \bar{D}_{\mathbf{k}} = \frac{i}{a} \left(\sum_{\mu=1}^4 \gamma'^\mu \cos ak_\mu \right) \quad ,\tag{2.6}$$

where $k_\mu = \mathbf{k} \cdot \boldsymbol{\mu}$. In the next subsection and in section 5, we will derive the explicit expression of these k_μ 's in terms of the weight vectors $\boldsymbol{\lambda}_l$ of the 5-dimensional representation of the $SU(5)$ symmetry as well as useful relations.

The zero modes of \mathcal{D}_{BC} are points in the reciprocal space; they are obtained by solving $\mathcal{D}_{BC} = 0$; which leads to the following condition

$$\sum_{\mu=1}^4 \gamma^\mu (\sin aK_\mu - \cos aK_\mu) - \Gamma \left(2 - \sum_{\mu=1}^4 \cos aK_\mu \right) = 0 \quad .\tag{2.7}$$

This condition is a constraint relation on the wave vector components K_μ ; it is solved by the two following wave vectors:

$$\begin{aligned}\text{point } K_{BC} &: K_1 = K_2 = K_3 = K_4 = 0 \quad , \\ \text{point } K'_{BC} &: K'_1 = K'_2 = K'_3 = K'_4 = \frac{\pi}{2a} \quad ,\end{aligned}\tag{2.8}$$

and are interpreted in lattice QCD as associated with the light quarks up and down. Notice that if giving up the γ'_μ - terms in eqs(2.1-2.3); i.e $\gamma'_\mu \rightarrow 0$, the remaining terms in \mathcal{D}_{BC} namely $D_{\mathbf{k}} \sim \gamma^\mu \sin aK_\mu$ have 16 zero modes given by the wave components $K_\mu = 0, \pi$. By switching on the γ'_μ -terms, 14 zeros are removed.

2.2 Beyond BC model

Although very important, the lattice action \mathcal{S}_{BC} is somehow very particular; it let suspecting to hide a more fundamental property which can be explicitly exhibited by using symmetries. This will be developed in details in next sections; but in due time, notice the three following features that can be understood as a way to exhibit the $SU(5)$ symmetry in BC model.

- 1) The price to pay for getting a Dirac operator with two zero modes is the involvement of the complexified Dirac matrices Υ^μ , $\tilde{\Upsilon}^\mu$ as well as the particular matrix Γ . Despite that it breaks the $SO(4)$ Lorentz symmetry since it can be written as

$$\Gamma = \frac{1}{2} \sum_{\mu=1}^4 \gamma^\mu v_\mu \quad (2.9)$$

with

$$v_\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.10)$$

violating explicitly the $SO(4)$, the matrix Γ plays an important role in studying the zero modes. The expression of the matrix Γ (2.2) should be thought of as associated precisely with the solution of the constraint relation

$$2\Gamma - \sum_{\mu=1}^4 \gamma^\mu = 0 \quad (2.11)$$

that is required by a hidden symmetry of the BC model namely the $SU(5)$ symmetry of the $4D$ hyperdiamond; see also the correspondence given in section 4 of ref [10].

- 2) the BC action \mathcal{S}_{BC} lives on a $4D$ lattice \mathbb{L}_4^{BC} generated by $\boldsymbol{\mu} \equiv \mathbf{v}_\mu$; i.e the vectors

$$\mathbf{v}_1 = \begin{pmatrix} v_1^x \\ v_1^y \\ v_1^z \\ v_1^t \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} v_2^x \\ v_2^y \\ v_2^z \\ v_2^t \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} v_3^x \\ v_3^y \\ v_3^z \\ v_3^t \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} v_4^x \\ v_4^y \\ v_4^z \\ v_4^t \end{pmatrix} \quad (2.12)$$

These $\boldsymbol{\mu}$ -vectors look somehow ambiguous to interpret by using the analogy with $4D$ graphene prototype; they are also ambiguous to interpret from the $SU(5)$ symmetry view. Indeed, to each site

$$\mathbf{r} \in \mathbb{L}_4^{BC},$$

there should be 5 first nearest neighbors that are rotated by $SU(5)$ symmetry. But from the BC action we learn that the first nearest neighbors to each site \mathbf{r} are:

$$\mathbf{r} \rightarrow \begin{cases} \mathbf{r} + a\mathbf{v}_1 \\ \mathbf{r} + a\mathbf{v}_2 \\ \mathbf{r} + a\mathbf{v}_3 \\ \mathbf{r} + a\mathbf{v}_4 \end{cases}, \quad (2.13)$$

The fifth missing one, namely

$$\mathbf{r} \rightarrow \mathbf{r} + a\mathbf{v}_5, \quad (2.14)$$

may be interpreted in the BC fermions as associated with the extra term involving the matrix Γ .

- 3) To take into account the five nearest neighbors, we then have to use the rigorous correspondence

$$\Gamma^\mu \rightarrow \mathbf{v}_\mu, \quad \text{and} \quad \Gamma^5 \rightarrow \mathbf{v}_5 \quad (2.15)$$

which can be also written in a combined form as follows

$$\Gamma^M \rightarrow \mathbf{v}_M, \quad (2.16)$$

with

$$\Gamma^M = (\Gamma^\mu, \Gamma^5) \quad \text{and} \quad \mathbf{v}_M = (\mathbf{v}_\mu, \mathbf{v}_5). \quad (2.17)$$

Because of the $SU(5)$ symmetry properties [9], we also have to require the condition

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = 0, \quad (2.18)$$

characterizing the 5 first nearest neighbors.

To determine the explicit expressions of the matrices Γ_M in terms of the usual Dirac ones, we modify the BC action (2.1) as follows

$$\mathcal{S}'_{BC} \sim \frac{1}{2a} \sum_{\mathbf{r}} \left(\sum_{M=1}^5 \bar{\Psi}_{\mathbf{r}} \Gamma^M \Psi_{\mathbf{r} + a\mathbf{v}_M} - \sum_{M=1}^5 \bar{\Psi}_{\mathbf{r} + a\mathbf{v}_M} \Gamma^M \Psi_{\mathbf{r}} \right), \quad (2.19)$$

exhibiting both $SO(4)$ and $SU(5)$ symmetries and leading to the following free Dirac operator

$$\begin{aligned} \mathcal{D} = & \frac{i}{2a} \sum_{\mu=1}^4 (\Gamma_\mu + \bar{\Gamma}_\mu) \sin(ak_\mu) + \frac{i}{2a} (\Gamma_5 + \bar{\Gamma}_5) \sin(ak_5) \\ & \frac{1}{2a} \sum_{\mu=1}^4 (\Gamma_\mu - \bar{\Gamma}_\mu) \cos(ak_\mu) + \frac{1}{2a} (\Gamma_5 - \bar{\Gamma}_5) \cos(ak_5) \end{aligned} \quad (2.20)$$

where $k_M = \mathbf{k} \cdot \mathbf{v}_M$ and where

$$\prod_{M=1}^5 e^{iak_M} = 1 \quad , \quad \sum_{M=1}^5 k_M = 0 \quad , \quad (2.21)$$

expressing the conservation of total momenta at each lattice site.

Equating with (2.4-2.5-2.6), we get the identities

$$\begin{aligned} \Upsilon_\mu + \bar{\Upsilon}_\mu &= \Gamma_\mu + \bar{\Gamma}_\mu \quad , \\ \Upsilon_\mu - \bar{\Upsilon}_\mu &= \Gamma_\mu - \bar{\Gamma}_\mu \quad , \end{aligned} \quad (2.22)$$

and

$$\frac{i}{2a} (\Gamma_5 + \bar{\Gamma}_5) \sin(ak_5) + \frac{1}{2a} (\Gamma_5 - \bar{\Gamma}_5) \cos(ak_5) = -\frac{4i}{2a} \Gamma \quad . \quad (2.23)$$

Eqs(2.22) are solved by $\Gamma_\mu = \Upsilon_\mu$; that is

$$\Gamma_\mu = \gamma^\mu + i\gamma'^\mu = \gamma^\mu + i(\Gamma - \gamma^\mu) \quad (2.24)$$

while

$$\begin{aligned} \Gamma_5 &= -2i\Gamma \quad \text{for} \quad \sin(ak_5) = 0 \quad , \\ \Gamma_5 &= -2\Gamma \quad \text{for} \quad \sin(ak_5) = 1 \quad . \end{aligned} \quad (2.25)$$

where $k_5 = -(k_1 + k_2 + k_3 + k_4)$.

In this 5-dimensional approach, to be further be developed in next sections, the ambiguity in dealing with the $\boldsymbol{\mu}$ -vectors is overcome; and the underlying $SO(4)$ and $SU(5)$ symmetries of the model in reciprocal space are explicitly exhibited.

2.3 KW fermions

The BC fermions described above is not the only model that has a Dirac operator with two zero modes. There is an other remarkable model that has been considered in recent literature [5, 6]. This is given by the so called *Karsten-Wilzeck* model [7, 8] whose free fermion lattice action reads as follows,

$$\begin{aligned} \mathcal{S}_{KW} \sim & \frac{1}{2a} \sum_{\mathbf{r}} \left(\sum_{\mu=1}^4 \bar{\Psi}_{\mathbf{r}} [\gamma^\mu - i\gamma_4 (1 - \delta_{\mu 4})] \Psi_{\mathbf{r}+\mathbf{a}\boldsymbol{\mu}} \right) + \sum_{\mathbf{r}} \bar{\Psi}_{\mathbf{r}} [m_0 + \frac{3i}{a}\gamma_4] \Psi_{\mathbf{r}} \\ & - \frac{1}{2a} \sum_{\mathbf{r}} \left(\sum_{\mu=1}^4 \bar{\Psi}_{\mathbf{r}+\mathbf{a}\boldsymbol{\mu}} [\gamma^\mu + i\gamma_4 (1 - \delta_{\mu 4})] \Psi_{\mathbf{r}} \right) . \end{aligned} \quad (2.26)$$

This is quite similar to (2.1); it is obtained by making the following substitutions

$$\gamma'_\mu \longrightarrow \gamma_4 (1 - \delta_{\mu 4}) \quad , \quad \Gamma \longrightarrow \frac{3i}{a} \gamma_4 \quad . \quad (2.27)$$

Moreover mapping the above lattice action to the reciprocal space, the Dirac operator \mathcal{D}_{KW} reads as, upon dropping the bare mass m_0 , as follows:

$$\mathcal{D}_{KW} = \sum_{\mu=1}^4 \gamma^\mu \sin a k_\mu + \frac{i}{a} \gamma^4 \sum_{\mu=1}^3 (1 - \cos a k_\mu) = 0. \quad (2.28)$$

This operator has also two zero modes located at the points K_{KW} and K'_{KW} ,

$$\begin{aligned} K_{KW} &= (0, 0, 0, 0) \\ K'_{KW} &= (0, 0, 0, \frac{\pi}{a}) \end{aligned} \quad (2.29)$$

The comments we gave in the case of BC fermions apply as well to this case. In particular, we have the *extended-KW* action

$$\mathcal{S}'_{KW} \sim \frac{1}{2a} \sum_{\mathbf{r}} \left(\sum_{M=1}^5 \bar{\Psi}_{\mathbf{r}} \Gamma^M \Psi_{\mathbf{r} + a \mathbf{v}_M} - \sum_{M=1}^5 \bar{\Psi}_{\mathbf{r} + a \mathbf{v}_M} \Gamma^M \Psi_{\mathbf{r}} \right) \quad , \quad (2.30)$$

where now

$$\begin{aligned} \Gamma_\mu &= \gamma^\mu + i \gamma_4 (1 - \delta_{\mu 4}) \quad , \\ \Gamma_5 &= -\frac{6i}{a} \gamma_4, \quad \text{for } \sin(a k_5) = 0 \quad , \\ \Gamma_5 &= -\frac{6}{a} \gamma_4, \quad \text{for } \sin(a k_5) = 1 \quad . \end{aligned} \quad (2.31)$$

In section 5, we will show that the BC and KW fermions as well as their extended versions with lattice actions \mathcal{S}'_{BC} and \mathcal{S}'_{KW} respectively given by (2.19)-(2.30), correspond in fact to particular realizations of the tensor Ω_μ^l of the proposal (1.1).

3 Hyperdiamond fermions

Before describing the proposal (1.1), it is interesting to start by recalling useful features of the hyperdiamond \mathbb{L}_4 [9, 10, 11].

3.1 the $4D$ lattice

The lattice \mathbb{L}_4 is a $4D$ lattice made by the superposition of two sublattices \mathcal{A} and \mathcal{B} with respective coordinates positions,

$$\begin{aligned} \frac{\mathbf{r}_A}{a} &= N_1 \boldsymbol{\alpha}_1 + N_2 \boldsymbol{\alpha}_2 + N_3 \boldsymbol{\alpha}_3 + N_4 \boldsymbol{\alpha}_4 \quad , \\ \mathbf{r}_B &= \mathbf{r}_A + d \frac{\sqrt{5}}{2} \boldsymbol{\lambda}_l \quad , \quad a = d \frac{\sqrt{5}}{2} \quad , \end{aligned} \quad (3.1)$$

where the N_i 's are arbitrary integers and the λ_l 's are the weight vectors of the 5-dimensional representation of $SU(5)$. For a related construction see [15, 16].

Recall that each λ_l weight has 4 components

$$\lambda_l^\mu = (\lambda_l^1, \lambda_l^2, \lambda_l^3, \lambda_l^4), \quad l = 0, 1, 2, 3, 4, \quad (3.2)$$

obeying the group representation identities

$$\lambda_0^\mu + \lambda_1^\mu + \lambda_2^\mu + \lambda_3^\mu + \lambda_4^\mu = 0 \quad , \quad \mu = 1, \dots, 4 \quad . \quad (3.3)$$

These identities can be interpreted physically in terms of the conservation of total momenta at each site of \mathbb{L}_4 .

$$\prod_{l=0}^4 \exp \left(\frac{ia}{\hbar} \sum_{\mu=1}^4 p_\mu \lambda_l^\mu \right) = \prod_{\mu=1}^4 \exp \left(\frac{ia}{\hbar} \sum_{l=0}^4 p_\mu \lambda_l^\mu \right) = 1 \quad . \quad (3.4)$$

Notice that the components λ_l^μ can be expressed in various, but equivalent, ways. Explicit ways were used in [9, 10, 11]; see also [12], but, to our understanding, the more convenient and significant way to do is the one using the $SU(5)$ simple roots as follows:

$$\begin{aligned} \lambda_0 &= +\frac{4}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4 \quad , \\ \lambda_1 &= -\frac{1}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4 \quad , \\ \lambda_2 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4 \quad , \\ \lambda_3 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 + \frac{1}{5}\alpha_4 \quad , \\ \lambda_4 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 - \frac{4}{5}\alpha_4 \quad . \end{aligned} \quad (3.5)$$

These vectors capture the fact that each site \mathbf{r}_A in the sublattice \mathcal{A} has 5 first nearest neighbors in the sublattice \mathcal{B} ,

$$\begin{array}{c|c} \text{one site} & \rightarrow \quad 5 \text{ first nearest neighbors} \\ \hline \mathbf{r}_A \in \mathcal{A} & \left\{ \begin{array}{l} \mathbf{r}_A + a\lambda_0 \\ \mathbf{r}_A + a\lambda_1 \\ \mathbf{r}_A + a\lambda_2 \\ \mathbf{r}_A + a\lambda_3 \\ \mathbf{r}_A + a\lambda_4 \end{array} \right. \in \mathcal{B} \end{array} \quad (3.6)$$

Notice also that the expression of the λ_l 's in terms of the roots is helpful not only for calculations; but also for exhibiting manifestly the role of the $SU(5)$ symmetry.

For later use, we also need the dual vectors,

$$\mathbf{G} = \frac{2\pi N_1}{a} \omega_1 + \frac{2\pi N_2}{a} \omega_2 + \frac{2\pi N_3}{a} \omega_3 + \frac{2\pi N_4}{a} \omega_4, \quad (3.7)$$

obeying the reciprocal lattice property

$$\exp(ia\mathbf{G}\cdot\mathbf{r}_A) = 1, \quad (3.8)$$

thanks to the duality relation $\boldsymbol{\omega}_i \cdot \boldsymbol{\alpha}_j = \delta_{ij}$.

3.2 the free model

The proposal (1.1) involves, in addition to the Dirac spinor $\Psi_{\mathbf{r}}$ and the γ_{μ} matrices, the remarkable complex tensor Ω_{μ}^l ; this is a bi-fundamental representation of $SO(4) \times SU(5)$ given by the complex 4×5 matrix,

$$\Omega_{\mu}^l = \begin{pmatrix} \Omega_1^0 & \Omega_1^1 & \Omega_1^2 & \Omega_1^3 & \Omega_1^4 \\ \Omega_2^0 & \Omega_2^1 & \Omega_2^2 & \Omega_2^3 & \Omega_2^4 \\ \Omega_3^0 & \Omega_3^1 & \Omega_3^2 & \Omega_3^3 & \Omega_3^4 \\ \Omega_4^0 & \Omega_4^1 & \Omega_4^2 & \Omega_4^3 & \Omega_4^4 \end{pmatrix}, \quad (3.9)$$

that allows to link the lattice Euclidean space time index μ , capturing a 4-component vector of $SO(4)$; and the index l of the 5-dimensional representation of the $SU(5)$ symmetry of the hyperdiamond. In some sense, it plays the role of γ'_{μ} and Γ in the BC model.

Notice that the action (1.1) can be also written in a similar form (2.19), as follows

$$\mathcal{S} \sim \frac{i}{4a} \sum_{\mathbf{r}} \left(\sum_{l=0}^4 \bar{\Psi}_{\mathbf{r}} \Gamma^l \Psi_{\mathbf{r}+a\boldsymbol{\lambda}_l} + \sum_{l=0}^4 \bar{\Psi}_{\mathbf{r}+a\boldsymbol{\lambda}_l} \bar{\Gamma}^l \Psi_{\mathbf{r}} \right) \quad (3.10)$$

where now the Γ^l has the following general form

$$\Gamma^l = \left(\sum_{\mu=1}^4 \gamma^{\mu} \Omega_{\mu}^l \right) \quad , \quad \bar{\Gamma}^l = \left(\sum_{\mu=1}^4 \gamma^{\mu} \bar{\Omega}_{\mu}^l \right) \quad . \quad (3.11)$$

In the reciprocal space, our massless free fermion lattice action reads as

$$\mathcal{S} \sim \sum_{\mathbf{k}} \left(\sum_{\mu=1}^4 \bar{\Psi}_{\mathbf{k}} \mathcal{D} \Psi_{\mathbf{k}} \right) \quad (3.12)$$

with Dirac operator given by

$$\mathcal{D} = \frac{i}{4a} \gamma^{\mu} (D_{\mu} + \bar{D}_{\mu}) \quad , \quad \frac{i}{4a} (D_{\mu} + \bar{D}_{\mu}) = \frac{1}{2} \text{Tr} (\gamma_{\mu} \mathcal{D}) \quad , \quad (3.13)$$

and where D_{μ} and its complex adjoint \bar{D}_{μ} read as follows:

$$D_{\mu} = \sum_{l=0}^4 \Omega_{\mu}^l e^{iak \cdot \boldsymbol{\lambda}_l} \quad , \quad \bar{D}_{\mu} = \sum_{l=0}^4 \bar{\Omega}_{\mu}^l e^{-iak \cdot \boldsymbol{\lambda}_l} \quad . \quad (3.14)$$

So the zero modes of the Dirac operator (3.13) are obtained by solving the following constraint equation

$$\text{Re} \left(\sum_{l=0}^4 \Omega_{\mu}^l e^{i\Phi_l} \right) = 0, \quad (3.15)$$

which reads explicitly like

$$\sum_{l=0}^4 [\Omega_{\mu}^l + \bar{\Omega}_{\mu}^l] \cos \Phi_l + i [\Omega_{\mu}^l - \bar{\Omega}_{\mu}^l] \sin \Phi_l = 0, \quad (3.16)$$

where Φ_l are the relative phases of the wave functions $\Psi_{\mathbf{r}+a\boldsymbol{\lambda}_l}$ with respect to $\Psi_{\mathbf{r}}$; they read in the first Brillouin zone as follows

$$\Phi_l = a(\mathbf{k} \cdot \boldsymbol{\lambda}_l) \quad , \quad 0 \leq \Phi_l < 2\pi, \quad (3.17)$$

with

$$\mathbf{k} \cdot \boldsymbol{\lambda}_l = \sum_{\mu=1}^4 k_{\mu} \cdot \lambda_l^{\mu} \quad . \quad (3.18)$$

to be denoted as $k_l \equiv \mathbf{k} \cdot \boldsymbol{\lambda}_l$. Moreover, because of the $SU(5)$ symmetry (3.3), we also have the constraint relation

$$\sum_{i=0}^4 \Phi_i = 0, \quad \text{mod } 2\pi \quad (3.19)$$

showing that the 5 phases Φ_l are related; so that one of them; say Φ_0 , can be expressed in terms of the others like

$$\Phi_0 = -(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4). \quad (3.20)$$

Notice that if solving the constraint eqs(3.16) as

$$\Phi_l = \frac{2\pi N}{5}, \quad N = 0, 1, 2, 3, 4, \quad (3.21)$$

we end with a constraint relation on Ω_{μ}^l namely

$$\sum_{l=0}^5 \text{Re}(\Omega_{\mu}^l) = 0. \quad (3.22)$$

This condition can be solved by using eq(3.3); this shows that $\text{Re}(\Omega_{\mu}^l)$ can be thought of as proportional to the weight vectors λ_l^{μ} of $SU(5)$. A particular realization of Ω_{μ}^l is given by the one considered in [9, 10, 11]; and which reads as follows

$$\Omega_{\mu}^l = \begin{pmatrix} 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{5} & -\sqrt{5} & -\sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} & -\sqrt{5} & \sqrt{5} & -\sqrt{5} \\ -4 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (3.23)$$

Notice in passing that the tensor $(\Omega_\mu^l)_{BC}$ and $(\Omega_\mu^l)_{KW}$ given in the introduction, and which are associated with the BC and KW models, obey as well the property (3.22); this feature will be proved in next section.

Below, we want to look for general solutions going beyond $\Phi_l = \frac{2\pi N}{5}$; and recover the special solutions (3.21) as particular cases. This also allows us to get more information on the role of the tensor Ω_μ^l .

4 Solutions with two zero modes

First we study a simple model with two zero modes having similar properties as the BC and KW ones; then we analyze the generic case involving the 4θ real moduli captured by the complex tensor Ω_μ^l ; and explore the link between the number of zero modes and the moduli space of Ω_μ^l .

4.1 Simple model

This lattice model is a simple toy prototype having a lattice Dirac operator $\mathcal{D}(\mathbf{k})$, of quite similar form to BC and KW operators considered in section 2, sharing with eq(1.1) the three following features:

- the operator $\mathcal{D}(\mathbf{k})$ has two zero modes located at points $\mathbf{K} = (K_1, K_2, K_3, K_4)$ and $\mathbf{K}' = (K'_1, K'_2, K'_3, K'_4)$ of the reciprocal space; that is: $\mathcal{D}(\mathbf{K}) = \mathcal{D}(\mathbf{K}') = 0$,
- the continuum limit of $\mathcal{D}(\mathbf{k})$ in the neighborhood of the points \mathbf{K} and \mathbf{K}' is given by the usual free Dirac operator $\sum_\mu \gamma^\mu k_\mu$ plus higher order terms that have an interpretation in the Symanzik effective theory.
- it has the $SU(5)$ symmetry of the hyperdiamond that permits to mimick tight binding model of $2D$ graphene; there the underlying $2D$ lattice is given by the honeycomb which is known to have an $SU(3)$ symmetry. The $SU(5)$ encountered in the present study is just the extension of the $SU(3)$ to $4D$ dimensions.

The lattice action \mathcal{S} of this model is of the form (1.1); but with a very particular tensor Ω_μ^l . It reads in the reciprocal space like

$$\mathcal{S} = \int \frac{d^4 k}{(2\pi)^4} \bar{\Psi}(\mathbf{k}) \mathcal{D}(\mathbf{k}) \Psi(\mathbf{k}). \quad (4.1)$$

For simplicity, we often write this action like $\sum_{\mathbf{k}} \bar{\Psi}_{\mathbf{k}} \mathcal{D}_{\mathbf{k}} \Psi_{\mathbf{k}}$ with the matrix operator $\mathcal{D}_{\mathbf{k}}$ given by the following 4×4 matrix,

$$\mathcal{D}_{\mathbf{k}} = \frac{-i}{a} \sum_{\mu=1}^4 \gamma^{\mu} \left[\sin a k_{\mu} - \frac{1}{1+\sqrt{2}} \left(\cos a k_{\mu} - v_{\mu} \cos \left(\sum_{\nu=1}^4 a k_{\nu} \right) \right) \right] \quad (4.2)$$

with the vector $v_{\mu} = (1, 1, 1, 1)$ as in eq(2.10). In addition to the *four* gamma matrices γ^{μ} , this operator involves also the particular combination $\sum_{\mu} \gamma^{\mu} v_{\mu}$ required by $SU(5)$ symmetry of the hyperdiamond. By comparing this matrix operator with the generic one (1.1) depending on the complex tensor $\Omega_{\mu}^l = R_{\mu}^l + i J_{\mu}^l$ namely,

$$\mathcal{D}_{\mathbf{k}, \Omega} = \frac{-i}{a} \sum_{\mu=1}^4 \gamma^{\mu} \left[\sum_{l=0}^4 J_{\mu}^l \sin a k_l - \sum_{l=0}^4 R_{\mu}^l \cos a k_l \right], \quad (4.3)$$

we find that (4.2) is indeed a particular operator of the general (4.3). The corresponding matrices are as follows,

$$R_{\mu}^l = \frac{1}{1+\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad J_{\mu}^l = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.4)$$

Notice that these matrices have no free degrees of freedom, since all parameters are fixed to defined numbers; they correspond to a particular point in the moduli space $\frac{U(4,5)}{U(4) \times U(5)}$ and obey the following remarkable properties

$$\sum_{l=0}^4 R_{\mu}^l = 0 \quad , \quad \sum_{\nu=1}^4 J_{\mu}^{\nu} = (1 + \sqrt{2}) R_{\mu}^0 \quad . \quad (4.5)$$

These two relations have a physical interpretation; they will be recovered in next subsubsection when we study the zero modes of the matrix operator $\mathcal{D}_{\mathbf{k}, \Omega}$ for generic Ω_{μ}^l . Eqs(4.5) are the conditions that should be obeyed by $\Omega_{\mu}^l = R_{\mu}^l + i J_{\mu}^l$ to have two zero modes located at

$$\begin{aligned} (K_{\mu}) &= \left(\frac{2\pi}{a}, \frac{2\pi}{a}, \frac{2\pi}{a}, \frac{2\pi}{a} \right) \quad , \quad K_0 = \frac{2\pi}{a}, \quad \text{mod } \frac{2\pi}{a} \\ (K_{\mu}) &= \left(\frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a} \right) \quad , \quad K_0 = \frac{-\pi}{a}, \quad \text{mod } \frac{2\pi}{a} \end{aligned} \quad (4.6)$$

with

$$K_0 = - \sum_{\mu=1}^4 K_{\mu} \quad , \quad K'_0 = - \sum_{\mu=1}^4 K'_{\mu} \quad (4.7)$$

Notice also that for $K_\mu = 0, \text{ mod } \frac{2\pi}{a}$; the operator $\mathcal{D}_{\mathbf{K}}$ vanishes exactly due to

$$\frac{i}{a(1+\sqrt{2})} \sum_{\mu=1}^4 \gamma^\mu (1-v_\mu) = 0, \quad (4.8)$$

and for $K'_\mu = \frac{\pi}{4a}, \text{ mod } \frac{2\pi}{a}$, it also vanishes because of the identity

$$\frac{-i}{a} \sum_{\mu} \gamma^\mu \left[\frac{\sqrt{2}}{2} - \frac{1}{1+\sqrt{2}} \left(\frac{\sqrt{2}}{2} + v_\mu \right) \right] = 0. \quad (4.9)$$

From these relations, one learns that the operator $\mathcal{D}_{\mathbf{K}}$ has, up to translations in the reciprocal space of the hyperdiamond, two zero modes located at the points \mathbf{K} and \mathbf{K}' . To see that these wave vectors belong indeed to the first Brillouin zone in the reciprocal space $\tilde{\mathcal{R}}^4$, we first use the expansions

$$\begin{aligned} \mathbf{K} &= Q_1 \boldsymbol{\omega}_1 + Q_2 \boldsymbol{\omega}_2 + Q_3 \boldsymbol{\omega}_3 + Q_4 \boldsymbol{\omega}_4 \quad , \\ \mathbf{K}' &= Q'_1 \boldsymbol{\omega}_1 + Q'_2 \boldsymbol{\omega}_2 + Q'_3 \boldsymbol{\omega}_3 + Q'_4 \boldsymbol{\omega}_4 \quad , \end{aligned} \quad (4.10)$$

with the Q_i 's and Q'_i 's real numbers; and where the $\boldsymbol{\omega}_i$'s are the four fundamental weight vectors of $\text{SU}(5)$ that generate $\tilde{\mathcal{R}}^4$; they obey amongst others eqs(3.7-3.8). Then, we compute the K_l 's and K'_l 's by help of the the relations (3.18) expressing these components in terms of the $\boldsymbol{\lambda}_l$'s respectively as $\mathbf{K} \cdot \boldsymbol{\lambda}_l$ and $\mathbf{K}' \cdot \boldsymbol{\lambda}_l$. Using eqs(3.5) giving the $\boldsymbol{\lambda}_l$'s in terms of the simple roots $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$; and the duality property $\boldsymbol{\omega}_i \cdot \boldsymbol{\alpha}_j = \delta_{ij}$, we can express the components $k_l = \mathbf{k} \cdot \boldsymbol{\lambda}_l$ as $\sum_i Q_i (\boldsymbol{\omega}_i \cdot \boldsymbol{\lambda}_l)$ or equivalently in matrix form like,

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & +\frac{3}{5} & +\frac{2}{5} & +\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & +\frac{2}{5} & +\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & -\frac{3}{5} & +\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & -\frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \quad (4.11)$$

This leads to

$$\begin{aligned} Q_1 &= -2k_1 - k_2 - k_3 - k_4 \\ Q_2 &= k_1 - k_2 \\ Q_3 &= k_2 - k_3 \\ Q_4 &= k_3 - k_4 \end{aligned} \quad (4.12)$$

From these relations we learn the two following: (i) the sum $k_1 + k_2 + k_3 + k_4 = -k_0$ is precisely $-\frac{1}{5} (4Q_1 + 3Q_2 + 2Q_3 + Q_4)$ that follows from the direct use of (3.5). (ii) In the first Brillouin zone given by $-\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a}$, the two Dirac point \mathbf{K} and \mathbf{K}' are located at

$$\mathbf{K} = 0 \quad , \quad \mathbf{K}' = \frac{3\pi}{4a} \boldsymbol{\omega}_1 = \left(\frac{\pi}{a} - \frac{\pi}{4a} \right) \boldsymbol{\omega}_1 \quad . \quad (4.13)$$

Notice that in the neighborhood of \mathbf{K} and \mathbf{K}' , the operator $\mathcal{D}_{\mathbf{q}+\mathbf{K}}$ and $\mathcal{D}_{\mathbf{q}+\mathbf{K}'}$ operators behave, up to the first order in the spacing parameter a , as follows

$$\begin{aligned}\mathcal{D}_{q+K} &= -i \sum_{\mu=1}^4 \gamma^\mu q_\mu - \frac{ia}{2(1+\sqrt{2})} \sum_{\mu=1}^4 \gamma^\mu [(q_\mu)^2 + v_\mu (\mathbf{v} \cdot \mathbf{q})^2] + O(a^2) \quad , \\ \mathcal{D}_{q+K'} &= -i \sum_{\mu=1}^4 \gamma^\mu q_\mu - \frac{ia}{2(1+\sqrt{2})} \sum_{\mu=1}^4 \gamma^\mu \left[\frac{\sqrt{2}}{2} (k_\mu)^2 - v_\mu (\mathbf{v} \cdot \mathbf{q})^2 \right] + O(a^2) \quad ,\end{aligned}\tag{4.14}$$

where

$$\mathbf{v} \cdot \mathbf{q} = \sum_{\nu} v^\nu q_\nu = q_1 + q_2 + q_3 + q_4.\tag{4.15}$$

These expansions in powers of the spacing parameter of the hyperdiamond give, at zero order $O(a^0)$, the usual Dirac operator of the continuum limit. At first order $O(a)$, we have 5-dimensional operators that play an important in the study of the Symanzik theory. The latter has a generic effective action that reads in powers of the parameter a as follows,

$$\mathcal{S}_{eff} = \frac{1}{a^4} \sum_n a^n \sum_j c_n^{(j)} \mathcal{O}_n^{(j)}.\tag{4.16}$$

Here, the $\mathcal{O}_n^{(j)}$'s are dimension n operators with relevant $\mathcal{O}_3^{(j)}$ and marginal $\mathcal{O}_4^{(j)}$ ones; some of these operators may emerge in the renormalization of the quantum theory. The last terms of (4.14) are linear in a and so lead to $\mathcal{O}_5^{(j)}$ operators; for explicit use of these operators; see [12].

4.2 Generic solutions

In this subsection we want to further explore the role of the complex tensor Ω_μ^l in the engineering of zero modes of the matrix operator $\mathcal{D}_{\mathbf{k},\Omega}$ of eq(1.1). We start from eq(4.3); then look for the interesting class of those $\mathcal{D}_{\mathbf{k},\Omega}$ operators having:

- two zero modes located at two different points \mathbf{K} and \mathbf{K}' of the reciprocal space. These modes, which should be non degenerate, require the two set of conditions:

$$\mathcal{D}_{\mathbf{K},\Omega} = 0 \quad \text{and} \quad \mathcal{D}_{\mathbf{K}',\Omega} = 0 \quad ,\tag{4.17}$$

- the appropriate continuum limit in neighborhood of K and K' given by the Dirac operator. From eq(4.3), this property requires amongst others that the imaginary part of Ω_μ^l should be as $J_\mu^l \neq A_\mu \otimes B^l$,
- the $SU(5)$ symmetry of the hyperdiamond to take advantage of known results on $2D$ graphene.

To avoid lengthy technical details on the engineering of the zero modes, we will come directly to the key points of our method of construction. The basic ideas are illustrated through explicit examples given in next sub-subsection; and general results will be given later.

4.2.1 Method of engineering zeros

First, notice that the values of the wave vector variable \mathbf{k} solving $\mathcal{D}_{\mathbf{k},\Omega} = 0$ depend on the components of Ω_μ^l ; this feature means that this zero mode equations can be also interpreted as constraint eqs relating wave vectors k_μ in the reciprocal lattice $\tilde{\mathcal{R}}^4$ to points in the real 40 - dimensional moduli space $\mathfrak{M}_{40} = \mathfrak{R}_{20} \times \mathfrak{R}_{20}$ given by,

$$\mathfrak{M}_{40} = \frac{SO(4,5)}{SO(4) \times SO(5)} \times \frac{SO(4,5)}{SO(4) \times SO(5)}. \quad (4.18)$$

The relations between the wave vectors \mathbf{k} and the moduli Ω_μ^l mean as well that we have the two following equivalent statements:

- Given some tensor Ω_μ^l , corresponding to a point in the moduli space \mathfrak{M}_{40} , the solving of $\mathcal{D}_{\mathbf{k},\Omega} = 0$ lead to wave vectors $\{K_\mu^{(1)}, K_\mu^{(2)}, \dots\}$ in the reciprocal space $\tilde{\mathcal{R}}^4$. This approach has been used in the above subsection with Ω_μ^l given by eqs(4.4).
- Inversely, given a wave vector K_μ , corresponding to point in $\tilde{\mathcal{R}}^4$, the matrix equation $\mathcal{D}_{\mathbf{k},\Omega} = 0$ allow to determine the set of points in \mathfrak{M}_{40} . In other words, $\mathcal{D}_{\mathbf{k},\Omega}$ should be thought of as constraint eqs on the 40 real moduli of the complex tensor $\Omega_\mu^l = R_\mu^l + iJ_\mu^l$.

Below, we develop this second approach.

Zero mode's constraint eqs and their linearization

To do so, notice that $\mathcal{D}_{\mathbf{k},\Omega} = 0$ is a 4×4 matrix operator equation that describe *a priori* a system of 16 eqs; but because of the properties of the Dirac matrices, only the 4 of these relations are relevant; they are given by,

$$Tr(\gamma_\mu \mathcal{D}_{\mathbf{k},\Omega}) = 0, \quad \mu = 1, 2, 3, 4. \quad (4.19)$$

Notice also that in the interesting case of two zero modes of $\mathcal{D}_{\mathbf{k},\Omega}$, we need to fix two points in the reciprocal space; say $\mathbf{K} = (K_1, K_2, K_3, K_4)$ and $\mathbf{K}' = (K'_1, K'_2, K'_3, K'_4)$ satisfying the conditions,

$$Tr(\gamma_\mu \mathcal{D}_{\mathbf{K},\Omega}) = 0, \quad Tr(\gamma_\mu \mathcal{D}_{\mathbf{K}',\Omega}) = 0. \quad (4.20)$$

These conditions constitute 8 constraints on the free parameters of the complex tensor Ω_μ^l ; their solutions require fixing 8 parameters of Ω_μ^l in terms of (K_1, K_2, K_3, K_4) and (K'_1, K'_2, K'_3, K'_4) ; and therefore lead to a reduction of the moduli space \mathfrak{M}_{40} down to some subspaces \mathfrak{M}_{32} ,

$$\mathfrak{M}_{40} \rightarrow \mathfrak{M}_{32}. \quad (4.21)$$

The mathematical structure of \mathfrak{M}_{32} depends obviously on the nature of the constraints on Ω_μ^l . To have an idea on the space \mathfrak{M}_{32} , we study below some explicit examples where \mathfrak{M}_{32} is, for instance, given by the complex coset group manifold $\frac{U(4,4)}{U(4) \times U(4)}$ or by the following real one $\frac{SO(4,3)}{SO(4) \times SO(3)} \times \frac{SO(4,5)}{SO(4) \times SO(5)}$. Recall by the way that $\dim U(N, M) = (N + M)^2$ and $\dim SO(N, M) = \frac{1}{2}(N + M)(M + N - 1)$.

Substituting the decomposition $\Omega_\mu^l = R_\mu^l + iJ_\mu^l$ back into (4.2), we get the following constraint relation for zero modes,

$$\mathcal{D}_{\mathbf{k}, \Omega} = \frac{-i}{a} \sum_{\mu=1}^4 \gamma^\mu \left(\sum_{l=0}^4 J_\mu^l \sin ak_l - \sum_{l=0}^4 R_\mu^l \cos ak_l \right) = 0, \quad (4.22)$$

where the $ak_l = \Phi_l$ are the phases of the wave propagation, along the λ_l direction in the reciprocal space, satisfying the $SU(5)$ symmetry condition,

$$\sum_{l=0}^4 k_l = 0, \mod \frac{2\pi}{a} \quad , \quad \sum_{l=0}^4 \Phi_l = 0, \mod 2\pi \quad . \quad (4.23)$$

Using the diagonal vector $\mathbf{v} = (1, 1, 1, 1)$, these constraints allow to express k_0 in terms of the others; and similarly for Φ_0 . We have:

$$k_0 = - \sum_{\mu=1}^4 k_\mu \equiv -\mathbf{k} \cdot \mathbf{v} \quad , \quad \Phi_0 = - \sum_{\mu=1}^4 \Phi_\mu \equiv -\Phi \cdot \mathbf{v} \quad . \quad (4.24)$$

Notice that eqs(4.22) is a system 4 equations with 4 variables $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and 40 real moduli given by the components of R_μ^l and J_μ^l . Moreover because of their dependence on the functions $\cos \Phi_\mu$ and $\sin \Phi_\mu$; and also because of the condition (4.23), these numerical eqs are highly non linear and therefore difficult to solve. To reduce this non linearity, it is interesting to work with the following new variables,

$$\begin{aligned} C_\mu &= \cos \Phi_\mu \quad , \quad S_\mu = \sin \Phi_\mu \quad , \\ \vartheta_\mu &= R_\mu^0 C_0 \quad , \quad \varphi_\mu = J_\mu^0 S_0 \quad , \end{aligned} \quad (4.25)$$

this allow to gain some simplicity into the zero mode constraint eqs. Focusing on the first Brillouin zone in the reciprocal space $-\frac{\pi}{a} \leq k_\mu \leq \frac{\pi}{a}$; and using the above change of variables we can put (4.22) into the form

$$\sum_{\nu=1}^4 (R_\mu^\nu C_\nu - J_\mu^\nu S_\nu) = J_\mu^0 S_0 - R_\mu^0 C_0 \quad , \quad (4.26)$$

but still with the non linear terms

$$C_0 = C_1 C_2 C_3 C_4 - C_1 C_2 S_3 S_4 - C_1 S_2 C_3 S_4 - C_1 S_2 S_3 C_4 - S_1 C_2 C_3 S_4 - S_1 C_2 S_3 C_4 - S_1 S_2 C_3 C_4 + S_1 S_2 S_3 S_4 \quad (4.27)$$

and similarly for others coming from the expansion of S_0 . We will see below how these non linearities can be overcome.

Solving eqs(4.17)

The solutions we are interested in here are those satisfying the 3 properties given in the begining of this subsection namely: (i) two zeros modes at \mathbf{K} and \mathbf{K}' of the reciprocal space, (ii) a Dirac operator in the continuum limit of the neighborhood of \mathbf{K} and \mathbf{K}' , and (iii) SU(5) symmetry.

Our method to solve eqs(4.17) involves 4 steps as follows:

- (1) Start from the generic zero mode constraint equations $\mathcal{D}_{\mathbf{k},\Omega} = 0$ given by (4.26),
- (2) Take two arbitrary wave vectors \mathbf{K} and \mathbf{K}' in the first Brillouin zone of the reciprocal space, which by help of eqs(4.25), can be related to C_l and S_l like,

$$e^{iaK \cdot \lambda_l} = C_l + iS_l \quad , \quad e^{iaK' \cdot \lambda_l} = C'_l + iS'_l \quad , \quad (4.28)$$

leading in turn to the following system of 4+4 relations defining the two zeros

$$\begin{aligned} \sum_{\nu=1}^4 (R_\mu^\nu C_\nu - J_\mu^\nu S_\nu) &= J_\mu^0 S_0 - R_\mu^0 C_0 \quad , \\ \sum_{\nu=1}^4 (R_\mu^\nu C'_\nu - J_\mu^\nu S'_\nu) &= J_\mu^0 S'_0 - R_\mu^0 C'_0 \quad . \end{aligned} \quad (4.29)$$

with variables C_ν and C'_ν (since $S_\nu = \pm\sqrt{1 - C_\nu^2}$, $S'_\nu = \pm\sqrt{1 - C'^2_\nu}$) and the free moduli R_μ^ν , R_μ^0 as well as J_μ^ν , J_μ^0 .

- (3) To surround the non linearities, we make a particular choice of K_μ and K'_μ in the reciprocal space and interpret (4.29) as constraint equations on the moduli. For example, one may take the wave vectors K_μ and K'_μ as follows,

$$K_\mu = (0, 0, 0, 0) \quad , \quad K'_\mu = \left(\frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a}\right) \quad . \quad (4.30)$$

These vectors are precisely the one that have been obtained in the previous subsection; but one can also consider other choices; some of them are described as remarks which will be given at the end of this construction. Using the above particular choice, we have the following:

$$\begin{aligned} K_0 = 0 \quad , \quad (C_0, S_0) = (+1, 0) \quad , \quad (C_\mu, S_\mu) = (1, 0) \\ K'_0 = -\frac{\pi}{a} \quad , \quad (C'_0, S'_0) = (-1, 0) \quad , \quad (C'_\mu, S'_\mu) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \end{aligned} \quad (4.31)$$

(4) Put these values back into eqs(4.29), we obtain the following constraint relations on the tensors R_μ^l and J_μ^l ,

$$R_\mu^0 + \sum_{\nu=1}^4 R_\mu^\nu = 0, \quad \sum_{\nu=1}^4 J_\mu^\nu = - (1 + \sqrt{2}) R_\mu^0, \quad (4.32)$$

which should be compared with (4.5). The first set of constraint eqs reduce the number of degrees of freedom of R_μ^l from 20 down to 16. The second set of constraints do the same thing; but with the tensor J_μ^l . So the moduli space $\mathfrak{M}_{40} = \mathfrak{R}_{20} \times \mathfrak{R}_{20}$ gets reduced down to the subspace

$$\mathfrak{M}_{32} = \frac{SO(4,4)}{SO(4) \times SO(4)} \times \frac{SO(4,4)}{SO(4) \times SO(4)}. \quad (4.33)$$

Therefore, the two zeros (4.30) live on \mathfrak{M}_{32} ; those missing 8 moduli living on the coset manifold $\mathfrak{M}_{40}/\mathfrak{M}_{32}$ has been freezed by eqs(4.30).

With this method, one can engineer various 4D lattice models living on hyperdiamond. In this regards, let us consider rapidly other examples; and turn after to give a general result through three claims.

A remark and other examples

The remark concerns the general form of the operator $\mathcal{D}_{\mathbf{k},\Omega}$ that has two zero modes located at the points $K_\mu = (0, 0, 0, 0)$ and $K'_\mu = (\frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a}, \frac{\pi}{4a})$. It is given by:

$$\begin{aligned} \mathcal{D}_{\mathbf{k},\Omega} = & \frac{-i}{a} \sum_{\mu=1}^4 \gamma^\mu \left(\sum_{\nu=1}^4 J_\mu^\nu \sin ak_\nu - \sum_{\nu=1}^4 R_\mu^\nu \cos ak_\nu \right) \\ & \frac{-i}{a} \sum_{\mu=1}^4 \gamma^\mu \left[-J_\mu^0 \sin \mathbf{ak} \cdot \mathbf{v} + \left(\sum_{\nu=1}^4 R_\mu^\nu \right) \cos \mathbf{ak} \cdot \mathbf{v} \right], \end{aligned} \quad (4.34)$$

with the condition

$$\sum_{\nu=1}^4 J_\mu^\nu = (1 + \sqrt{2}) \sum_{\nu=1}^4 R_\mu^\nu. \quad (4.35)$$

Notice that in the neighborhood of K_μ , the expansion of $\mathcal{D}_{\mathbf{K}+\mathbf{q},\Omega}$ reads, up to second order in the power of the lattice spacing parameter a , like:

$$\begin{aligned} \mathcal{D}_{\mathbf{q},\Omega} = & -i \sum_{\mu=1}^4 \gamma^\mu \left(\sum_{\nu=1}^4 [J_\mu^\nu - v^\nu J_\mu^0] q_\nu \right) \\ & \frac{-ai}{2} \sum_{\mu=1}^4 \gamma^\mu \left[\sum_{\nu=1}^4 (q_\nu)^2 R_\mu^\nu + (\mathbf{q} \cdot \mathbf{v})^2 R_\mu^0 \right] + O(a^2), \end{aligned} \quad (4.36)$$

where we have used $\sum_{\nu=1}^4 R_\mu^\nu = -R_\mu^0$. To interpret q_μ as the real wave vector of the Dirac theory, we have to require moreover

$$J_\mu^\nu - v^\nu J_\mu^0 \sim \delta_\mu^\nu. \quad (4.37)$$

The other examples concern different kinds of two zero modes other than the ones given by (4.30). For instance, in the case where the two zero modes are chosen like in BC model (2.8) namely

$$\begin{aligned}\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 = 0 \quad , \quad \Phi_0 = 0 \quad , \\ \Phi'_1 = \Phi'_2 = \Phi'_3 = \Phi'_4 = \frac{\pi}{2} \quad , \quad \Phi'_0 = -2\pi \quad ,\end{aligned}\tag{4.38}$$

the analogue of eqs(4.31) read like,

$$\begin{aligned}K_0 = 0 \quad , \quad (C_0, S_0) = (1, 0) \quad , \quad (C_\mu, S_\mu) = (1, 0) \\ K'_0 = -\frac{2\pi}{a} \quad , \quad (C'_0, S'_0) = (1, 0) \quad , \quad (C'_\mu, S'_\mu) = (0, 1)\end{aligned}\tag{4.39}$$

Putting these values back into (4.29), we get the following constraint relations,

$$\sum_{\nu=1}^4 R_\mu^\nu = -R_\mu^0 \quad , \quad \sum_{\nu=1}^4 J_\mu^\nu = R_\mu^0 \quad ,\tag{4.40}$$

which can be also put into the form

$$\sum_{\nu=1}^4 (R_\mu^\nu + iJ_\mu^\nu) = (-1 + i) R_\mu^0.\tag{4.41}$$

These condition are manifestly satisfied by the Boriçi-Creutz matrix given by eq(1.2) and the moduli space of the tensor Ω_μ^l is given by $\frac{U(4,4)}{U(4) \times U(4)}$. Similarly, in the case the two zero modes are chosen like

$$\begin{aligned}\Phi_1 = \Phi_2 = \Phi_3 = 0, \quad \Phi_4 = 0, \quad \Phi_0 = 0 \quad , \\ \Phi'_1 = \Phi'_2 = \Phi'_3 = 0, \quad \Phi'_4 = \pi, \quad \Phi'_0 = -\pi \quad ,\end{aligned}\tag{4.42}$$

as in the KW fermions, the analogue of eqs(4.31) read like,

$$\begin{aligned}K_0 = 0 \quad , \quad (C_0, S_0) = (+1, 0) \quad , \quad (C_4, S_4) = (+1, 0) \\ K'_0 = 0 \quad , \quad (C'_0, S'_0) = (-1, 0) \quad , \quad (C'_4, S'_4) = (-1, 0) \quad ,\end{aligned}\tag{4.43}$$

and for $\mu = 1, 2, 3$, we have

$$\begin{aligned}(C_\mu, S_\mu) = (1, 0) \quad , \\ (C'_\mu, S'_\mu) = (1, 0) \quad .\end{aligned}\tag{4.44}$$

The constraint relations (4.29) become,

$$R_\mu^0 + \sum_{\nu=1}^4 R_\mu^\nu = 0 \quad , \quad R_\mu^1 + R_\mu^2 + R_\mu^3 = R_\mu^0 + R_\mu^4 \quad ,\tag{4.45}$$

exactly as for the KW matrix given by eq(1.3); see also next section. Notice that, in this example, the $4+4$ constraint relations are all of them imposed on R_μ^l , the real part of Ω_μ^l ; so the moduli space associated with these classes of solutions is given by

$$\mathfrak{M}'_{32} = \frac{SO(4, 3)}{SO(4) \times SO(3)} \times \frac{SO(4, 5)}{SO(4) \times SO(5)}\tag{4.46}$$

Below, we give three claims summarizing the basic results following from this construction.

4.2.2 three claims

Claim 1 : Moduli space of $\mathcal{D}_{\mathbf{k},\Omega}$

The moduli space of the operator $\mathcal{D}_{\mathbf{k},\Omega} \sim \frac{1}{a} \sum_{\mu} \gamma^{\mu} (\sum_l \Omega_{\mu}^l e^{ia\mathbf{k} \cdot \boldsymbol{\lambda}_l} + hc)$ on 4D hyperdiamond is given by the complexification of the real 20-dimensional space $\mathfrak{R}_{20} = \frac{SO(4,5)}{SO(4) \times SO(5)}$. Each copy of the two \mathfrak{R}_{20} 's is associated with the real R_{μ}^l and imaginary J_{μ}^l parts of the complex tensor Ω_{μ}^l . The combination of these spaces leads to the following complex 20-dimensional coset group manifold

$$\mathfrak{C}_{20} = \frac{U(4,5)}{U(4) \times U(5)} \sim \mathfrak{M}_{40} \quad (4.47)$$

Claim 2 : Engineering zero modes

Each zero mode of $\mathcal{D}_{\mathbf{k},\Omega}$, located at some wave vector point $\mathbf{K} = (K_1, K_2, K_3, K_4)$ in the reciprocal space of the 4D hyperdiamond, requires four real constraint relations given by the vanishing conditions $\text{Tr} [\gamma_{\mu} \mathcal{D}_{\mathbf{k},\Omega}] = 0$. These constraints reduce the moduli space \mathfrak{M}_{40} down to some submanifolds \mathfrak{M}_{36} . Similarly, two zero modes, located at two points \mathbf{K} and \mathbf{K}' in the reciprocal space, require 4 complex (4 + 4 real) conditions given by

$$\text{Tr} [\gamma_{\mu} \mathcal{D}_{\mathbf{K},\Omega}] = 0 \quad , \quad \text{Tr} [\gamma_{\mu} \mathcal{D}_{\mathbf{K}',\Omega}] = 0 \quad . \quad (4.48)$$

These constraint relations reduce the moduli space \mathfrak{M}_{40} down to real 32-dimensional subspaces \mathfrak{M}_{32} . In the case of BC fermions, the \mathfrak{M}_{32} manifold is given by

$$\mathfrak{C}_{16} = \frac{U(4,4)}{U(4) \times U(4)}, \quad (4.49)$$

and in KW fermions; it reads as $\frac{SO(4,3)}{SO(4) \times SO(3)} \times \frac{SO(4,5)}{SO(4) \times SO(5)}$. For the generic case of $(n+m)$ zero modes of $\mathcal{D}_{\mathbf{K},\Omega}$, one should have $4(n+m)$ conditions; and the corresponding moduli spaces have the form $\mathfrak{M}_{40-4n-4m}$. Typical examples of these submanifolds are given by $\mathfrak{R}_{20-4n} \times \mathfrak{R}_{20-4m}$ with $\mathfrak{R}_{20-4s} = \frac{SO(4,5-s)}{SO(4) \times SO(5-s)}$.

Claim 3 : Continuum limit of $\mathcal{D}_{q+K,\Omega}$

In the neighborhood of each zero mode K with $\text{Tr} [\gamma_{\mu} \mathcal{D}_{\mathbf{K},\Omega}] = 0$, the operator $\mathcal{D}_{q+K,\Omega}$ behaves, at first order in the lattice spacing parameter a , like

$$\begin{aligned} \mathcal{D}_{q+K,\Omega} \sim & - \sum_{\mu=1}^4 \gamma^{\mu} p_{\mu} \\ & - a \sum_{\mu=1}^4 \gamma^{\mu} [R_{\mu}^l \cos aK_l - J_{\mu}^l \sin aK_l] (q_l)^2 + O(a^2) \end{aligned} \quad (4.50)$$

where the wave vector p_{μ} is related to q_l as follows

$$p_{\mu} = \sum_{\nu=1}^4 (R_{\mu}^{\nu} S_{\nu} + J_{\mu}^{\nu} S_{\nu}) q_{\nu} + (R_{\mu}^0 S_0 + J_{\mu}^0 C_0) q_0, \quad (4.51)$$

with $q_l = \mathbf{q} \cdot \boldsymbol{\lambda}_l$, $K_l = \mathbf{K} \cdot \boldsymbol{\lambda}_l$ and $C_l = \cos aK_l$, $S_l = \sin aK_l$. To interpret the q_μ 's as the real wave vector components of the continuum limit, we have to impose moreover $(R_\mu^\nu S_\nu + J_\mu^\nu C_l) \sim \delta_\mu^\nu$ and $(R_\mu^0 S_0 + J_\mu^0 C_0) = 0$.

5 Re-deriving BC and KW fermions

In this section, we use $SU(5)$ symmetry of the hyperdiamond to re-derive the BC fermions and the KW ones from our general action. We first consider BC model and then the KW fermions

5.1 BC fermions

A simple way to re-derive the BC fermions from our proposal is to work in the reciprocal space and compare the Dirac operators following from the models (2.1) and (1.1). In the case of the BC fermions, the Dirac operator \mathcal{D}_{BC} reads, up to a scale factor and upon dropping the mass term, as follows

$$\mathcal{D}_{BC} \sim \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu \sin ap_\mu - \frac{2i}{a} \Gamma - \frac{i}{a} \sum_{\mu=1}^4 (\gamma^\mu - \Gamma) \cos ap_\mu, \quad (5.1)$$

with

$$\Gamma = \frac{1}{2} (\gamma^1 + \gamma^2 + \gamma^3 + \gamma^4) \quad , \quad (5.2)$$

which we rewrite as

$$\Gamma = \frac{1}{2} \sum_{\mu=1}^4 \gamma^\mu v_\mu \quad , \quad v_\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad . \quad (5.3)$$

In our proposal (1.1), the Dirac operator reads as

$$\begin{aligned} \mathcal{D} \sim & \sum_{\mu=1}^4 \gamma^\mu [\Omega_\mu^0 - \bar{\Omega}_\mu^0] \sin ap_0 + \sum_{\mu,\nu=1}^4 \gamma^\mu [\Omega_\mu^\nu - \bar{\Omega}_\mu^\nu] \sin ap_\nu \\ & - i \sum_{\mu=1}^4 \gamma^\mu [\Omega_\mu^0 + \bar{\Omega}_\mu^0] \cos ap_0 - i \sum_{\mu,\nu=1}^4 \gamma^\mu [\Omega_\mu^\nu + \bar{\Omega}_\mu^\nu] \cos ap_\nu. \end{aligned} \quad (5.4)$$

By equating (5.1) and (5.4), one gets the tensor Ω_μ^l that define the BC model. Taking the components of the tensor Ω_μ^l as follows

$$\Omega_\mu^l \neq \bar{\Omega}_\mu^l \quad (5.5)$$

with

$$\Omega_\mu^0 = \frac{1}{2}\xi_\mu \quad , \quad \Omega_\mu^\nu = \frac{1}{2}(1+i)\delta_\mu^\nu - \frac{1}{4}\Sigma_\mu^\nu \quad , \quad (5.6)$$

and putting back into (5.4), we first get

$$\begin{aligned} \mathcal{D} = & \frac{1}{2a} \sum_{\mu=1}^4 \gamma^\mu [\xi_\mu - \bar{\xi}_\mu] \sin ap_0 + \frac{i}{2a} \sum_{\mu=1}^4 \gamma^\mu \sin ap_\mu \\ & - \frac{i}{2a} \sum_{\mu=1}^4 \gamma^\mu [\xi_\mu + \bar{\xi}_\mu] \cos ap_0 - \frac{i}{2a} \sum_{\mu,\nu=1}^4 \gamma^\mu [2\delta_\mu^\nu - \Sigma_\mu^\nu] \cos ap_\nu. \end{aligned} \quad (5.7)$$

Then equating with (5.1), we obtain:

$$\xi_\mu - \bar{\xi}_\mu = 0 \quad , \quad \text{or} \quad \sin ap_0 = 0 \quad , \quad (5.8)$$

and

$$\begin{aligned} \Omega_\mu^\nu - \bar{\Omega}_\mu^\nu &= i\delta_\mu^\nu \quad , \\ \Omega_\mu^\nu + \bar{\Omega}_\mu^\nu &= \delta_\mu^\nu - \frac{1}{2}\Sigma_\mu^\nu \quad , \end{aligned} \quad (5.9)$$

where ξ_μ and Σ_μ^ν are still to determine. Solving these quantities as follows,

$$\xi_\mu = v_\mu \quad , \quad \Sigma_\mu^\nu = v_\mu \otimes v^\nu \quad , \quad (5.10)$$

where v_μ is as in (5.3) and where

$$\Sigma_\mu^\nu = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad (5.11)$$

then putting back into (5.7), we end with the following factorized form

$$\begin{aligned} \mathcal{D} = & i \sum_{\mu=1}^4 \gamma^\mu \sin ap_\mu - i \cos ap_0 \left(\sum_{\mu=1}^4 \gamma^\mu v_\mu \right) - i \sum_{\mu=1}^4 \gamma^\mu \cos ap_\mu \\ & + \frac{i}{2} \left(\sum_{\mu=1}^4 \gamma^\mu v_\mu \right) \left(\sum_{\nu=1}^4 v^\nu \cos ap_\nu \right). \end{aligned} \quad (5.12)$$

Using (5.3), we obtain

$$\begin{aligned} \mathcal{D} = & \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu \sin ap_\mu - \frac{2i}{a} \Gamma \cos ap_0 \\ & + \frac{i}{a} \Gamma \sum_{\nu=1}^4 \cos ap_\nu - \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu \cos ap_\mu. \end{aligned} \quad (5.13)$$

But this operator is precisely the one given by (2.20) with $\Gamma_5 = -2i\Gamma$ and $ap_5 = ap_0$. Comparing with (5.1), we discover that we should have

$$\cos ap_0 = 1 \implies p_0 = 0, \mod\left(\frac{2\pi}{a}\right) . \quad (5.14)$$

Let us discuss the meaning of these constraint equations. In our proposal, the components (p_0, p_μ) form a 5-dimensional vector p_l given by

$$p_l = \sum_{\mu=1}^4 k_\mu \cdot \lambda_l^\mu \quad , \quad l = 0, \dots, 4 \quad , \quad (5.15)$$

where λ_l^μ are given by eqs(3.2-3.5). Because $(\lambda_0^\mu + \lambda_1^\mu + \lambda_2^\mu + \lambda_3^\mu + \lambda_4^\mu) = 0$, these momenta satisfy the conservation law

$$p_0 = -(p_1 + p_2 + p_3 + p_4) . \quad (5.16)$$

Now using the fact that the zero modes of the Dirac operator of the BC fermions are either $p_1 = p_2 = p_3 = p_4 = 0$ or $p_1 = p_2 = p_3 = p_4 = \frac{\pi}{a}$, one finds indeed that $p_0 = 0, \mod\frac{2\pi}{a}$.

5.2 KW fermions

In the KW fermions, the Dirac operator \mathcal{D}_{KW} is given by eq(2.28). By dropping the bare mass m_0 , it reads, up to a global scale factor, as follows

$$\mathcal{D}_{KW} \sim \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu \sin ak_\mu - \frac{i}{a} \gamma_4 \sum_{\mu=1}^3 \cos ak_\mu + \frac{3}{a} i \gamma_4 . \quad (5.17)$$

To make contact between this 4×4 matrix operator and the \mathcal{D} one following from our proposal, we should solve the equation $\mathcal{D}_{KW} = \mathcal{D}$ with ,

$$\begin{aligned} \mathcal{D} \sim & \frac{1}{a} \sum_{\mu=1}^4 \gamma^\mu [\Omega_\mu^0 - \bar{\Omega}_\mu^0] \sin ap_0 + \frac{1}{a} \sum_{\mu, \nu=1}^4 \gamma^\mu [\Omega_\mu^\nu - \bar{\Omega}_\mu^\nu] \sin ap_\nu \\ & - \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu [\Omega_\mu^0 + \bar{\Omega}_\mu^0] \cos ap_0 - \frac{i}{a} \sum_{\mu, \nu=1}^4 \gamma^\mu [\Omega_\mu^\nu + \bar{\Omega}_\mu^\nu] \cos ap_\nu . \end{aligned} \quad (5.18)$$

By choosing the Ω_μ^l components as follows,

$$\Omega_\mu^l \neq \bar{\Omega}_\mu^l \quad (5.19)$$

with

$$\Omega_\mu^0 = \zeta_\mu \quad , \quad \Omega_\mu^\nu = \frac{i}{2} \delta_\mu^\nu + \frac{1}{2} \Theta_\mu^\nu \quad , \quad (5.20)$$

or equivalently

$$\Omega_\mu^l = \frac{1}{2} \begin{pmatrix} 2\zeta_1 & i + \Theta_1^1 & \Theta_2^1 & \Theta_3^1 & \Theta_4^1 \\ 2\zeta_2 & \Theta_1^2 & i + \Theta_2^2 & \Theta_3^2 & \Theta_4^2 \\ 2\zeta_3 & \Theta_1^3 & \Theta_2^3 & i + \Theta_3^3 & \Theta_4^3 \\ 2\zeta_4 & \Theta_1^4 & \Theta_2^4 & i + \Theta_3^4 & \Theta_4^4 \end{pmatrix}, \quad (5.21)$$

where ζ_μ and Θ_μ^ν real. Putting back into (5.18), we get the reduced operator,

$$\begin{aligned} \mathcal{D} \sim & \frac{i}{a} \sum_{\mu=1}^4 \gamma^\mu \sin ap_\mu - \frac{2i}{a} \cos ap_0 \left(\sum_{\mu=1}^4 \gamma^\mu \zeta_\mu \right) \\ & - \frac{i}{a} \sum_{\nu=1}^4 \left(\sum_{\mu=1}^4 \gamma^\mu \Theta_\mu^\nu \right) \cos ap_\nu. \end{aligned} \quad (5.22)$$

Comparing with (5.17), it follows that the choice

$$\zeta_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_0 \end{pmatrix}, \quad \Theta_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}, \quad (5.23)$$

leads to

$$\begin{aligned} \mathcal{D} \sim & \frac{i}{a} \sum_{\mu=1}^4 \gamma_\mu \sin ap_\mu - \gamma_4 \left[\frac{2i}{a} c_0 \cos ap_0 \right] \\ & - \gamma_4 \left(\frac{i}{a} \sum_{\mu=1}^4 c_\mu \cos ap_\mu \right). \end{aligned} \quad (5.24)$$

By requiring the KW zeros

$$\begin{aligned} (1) : & \quad p_1 = p_2 = p_3 = 0 \quad , \quad p_4 = p_0 = 0 \quad , \\ (2) : & \quad p_1 = p_2 = p_3 = 0 \quad , \quad p_4 = p_0 = \pi \quad , \end{aligned} \quad (5.25)$$

obeying $\sum_{l=0}^4 p_l = 0 \bmod (2\pi)$, we obtain the following constraints on the coefficients

$$\begin{aligned} (1) : & \quad c_1 + c_2 + c_3 + c_4 + 2c_0 = 0 \quad , \\ (2) : & \quad c_1 + c_2 + c_3 - c_4 - 2c_0 = 0 \quad . \end{aligned} \quad (5.26)$$

A particular solution of these relations is given by

$$c_1 = c_2 = c_0 \quad , \quad c_3 = c_4 = -2c_0 \quad , \quad c_0 = 1. \quad (5.27)$$

6 Conclusion and comments

Motivated by studies on lattice QCD and using results on $4D$ graphene, we have developed in this paper a class of $4D$ lattice fermions that live on the hyperdiamond \mathbb{L}_4 with typical action given by (1.1). This lattice action involves a complex tensor Ω_μ^l that captures basic properties on hyperdiamond fermions; in particular:

- (1) it links the $4D$ Euclidean vector μ and the 5-dimensional λ_l allowing to exhibit explicit both the underlying $SO(4)$ and $SU(5)$ symmetries of the lattice action,
- (2) it encodes the data on the zero modes of the Dirac operator \mathcal{D} . The constraint relation giving the zero modes of the Dirac operator reads, in terms of the wave vector \mathbf{K} , as follows:

$$\text{Re} \left(\sum_{l=0}^4 \Omega_\mu^l e^{ia\mathbf{K} \cdot \lambda_l} \right) = 0 \quad (6.1)$$

and its solutions depend indeed on Ω_μ^l and the weight vectors λ_l ,

- (3) it contains as particular examples the BC and KW fermions; the corresponding $(\Omega_\mu^l)_{BC}$ and $(\Omega_\mu^l)_{KW}$ were given in the introduction.

We end this study by noting that one can use the results developed in this paper to investigate the renormalization properties of the gauged version of (1.1) along the same lines as done in [5, 6]. For instance, the free quark propagator following from our proposal is given by

$$\mathbf{S}(p) \sim a \frac{\gamma^\mu (D_\mu - \bar{D}_\mu) + aM_0}{D^2 - \bar{D}^2 + (aM_0)^2}, \quad (6.2)$$

with bare mass M_0 ; and

$$D_\mu = \sum_{l=0}^4 \Omega_\mu^l e^{ia\mathbf{p} \cdot \lambda_l} \quad , \quad \bar{D}_\mu = \sum_{l=0}^4 \bar{\Omega}_\mu^l e^{-ia\mathbf{p} \cdot \lambda_l} \quad , \quad (6.3)$$

and where $D^2 = D_\mu D^\mu$, $\bar{D}^2 = \bar{D}_\mu \bar{D}^\mu$. The gauging of (1.1) may be done by making Ω_μ^l a local field; i.e by allowing it to depend on the position. Progress in this direction will be given in a future occasion.

Acknowledgement 4 *the authors would like to thank M.Bousmina for discussions; Tatsuhiro Misumi and Simon Catterall for drawing our attention to their works [14, 15, 16]. L.B Drissi would like to thank ICTP for associate membership and E.H Saidi thanks URAC-09, CNRST.*

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